Formulation of the STT Track Fit

DØ Note xxxx

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Abstract

The trajectory of a charged particle in the plane transverse to a magnetic field is a circle in the absence of multiple scattering. For particles which originate near the origin, the circular path can be written as

$$\phi(r) = \frac{b}{r} + \kappa r + \phi_0 \tag{1}$$

in which ϕ is the azimuthal angle of the point on the track at radius r, b is the distance-of-closest-approach of the track to the origin, ϕ_0 is the angle of the track at the point-of-closest-approach and $\kappa \equiv 1/(2R)$ is the track curvature. Both b and κ are signed. This form represents an approximation to the circle under the condition $\kappa r \ll 1.0$, $br \ll 1.0$ and $\phi - \phi_0 \ll 1.0$. For DØ this approximation holds for $|b| < 2$ mm and for all relevent transverse momenta.

This form can be used in a χ^2 minimization to find the parameters of a given charged particle track. The χ^2 is given in general by

$$\chi^2 = \sum_{hits} \Big[\frac{(\phi(r_i) - \phi_i)}{\sigma_{\phi_i}}\Big]^2$$

in which (r_i, ϕ_i) are reconstructed hits in the DØ silicon and fiber tracking detectors and σ_{ϕ_i} is the error in ϕ. A more convenient form involves not the angular hit resolution, but the resolution σ_x measured in the plane approximately perpendicular to the track trajectory. In this form, $\sigma_{\phi_i} \equiv \sigma_x / r_i$. This form is useful because one needs only two resolution values σ_{CFT} and σ_{SMT} for hits from the fiber and silicon detectors respectively [1] In this case the χ^2 becomes (after substituting eqn. 1)

$$\chi^2 = \sum_{hits} \Big[\frac{b + \kappa r_i^2 + \phi_0 r_i - \phi_i r_i}{\sigma_{xi}}\Big]^2 \tag{2}$$

[1] For the most precise results, hit-by-hit resolutions should be used. However, given the initial approximation, the method used here holds.

The best fit to the track parameters is found by minimizing this χ^2 with respect to b, κ and ϕ_0 in the usual manner. Because $\phi(r)$ is linear in the parameters, this can be performed analytically. Taking the derivative of eqn. 2 with respect to each track parameter gives:

$$\begin{aligned}
\frac{1}{2}\frac{\partial\chi^2}{\partial b} &= \sum_{hits}\Big(\frac{b+\kappa r_i^2+\phi_0 r_i-\phi_i r_i}{\sigma_{xi}^2}\Big) \\
\frac{1}{2}\frac{\partial\chi^2}{\partial \phi_0} &= \sum_{hits} r_i\Big(\frac{b+\kappa r_i^2+\phi_0 r_i-\phi_i r_i}{\sigma_{xi}^2}\Big) \\
\frac{1}{2}\frac{\partial\chi^2}{\partial \kappa} &= \sum_{hits} r_i^2\Big(\frac{b+\kappa r_i^2+\phi_0 r_i-\phi_i r_i}{\sigma_{xi}^2}\Big)
\end{aligned}$$

If we define $c_{ij} \equiv r_i^j/\sigma_i^2$ and $\Phi_{ij} \equiv \phi_i r_i^j/\sigma_i^2$, $c_j \equiv \sum_i c_{ij}$ and $\Phi_j \equiv \sum_i \Phi_{ij}$ and rearrange terms, the above equations reduce to the linear system

$$\mathbf{\Phi} = \mathbf{Cp}$$

with

$$\mathbf{p} \equiv \begin{pmatrix} b \\ \phi_0 \\ \kappa \end{pmatrix}, \quad \mathbf{C} \equiv \begin{pmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix} \text{ and } \mathbf{\Phi} \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}$$

This can be solved in the usual manner, giving $\mathbf{p} = \mathbf{C^{-1}\Phi}$. For completeness, the inverse is

$$\mathbf{C^{-1}} = \frac{1}{D} * \begin{pmatrix} c_4c_2 - c_3^2 & c_2c_3 - c_1c_4 & c_1c_3 - c_2^2 \\ c_2c_3 - c_1c_4 & c_0c_4 - c_2^2 & c_1c_2 - c_0c_3 \\ c_1c_3 - c_2^2 & c_1c_2 - c_0c_3 & c_0c_2 - c_1^2 \end{pmatrix}$$

with the determinant given by $D = c_0c_2c_4 + 2c_1c_2c_3 - c_0c_3^2 - c_2^3 - c_1^2c_4$. When computing the inverse only the upper triangular terms are computed, and the lower half is obtained by the symmetry condition. The resulting track parameters are given by

$$b = \frac{1}{D}\Big[(c_4c_2 - c_3^2)\Phi_1 + (c_2c_3 - c_1c_4)\Phi_2 + (c_1c_3 - c_2^2)\Phi_3\Big] \tag{3}$$

$$\phi_0 = \frac{1}{D}\Big[(c_2c_3 - c_1c_4)\Phi_1 + (c_0c_4 - c_2^2)\Phi_2 + (c_1c_2 - c_0c_3)\Phi_3\Big] \tag{4}$$

$$\kappa = \frac{1}{D}\Big[(c_1c_3 - c_2^2)\Phi_1 + (c_1c_2 - c_0c_3)\Phi_2 + (c_0c_2 - c_1^2)\Phi_3\Big] \tag{5}$$

and the χ^2 is found by substituting the parameters back into eqn. 2 and rearranging to get

$$\chi^2 = \sum_{i=hits} c_{i0}[b + (\kappa r_i + \phi_0 - \phi_i)r_i]^2 \tag{6}$$

1 Alternate Format

The above method for writing the linear equation is the most compact. However, it does not have the best numerical properties. The elements of the Φ vector have different dimensions

which leads to very different numerical scales for each of the terms in both Φ and the inverse matrix. Because the Φ vector is linear in ϕ_i, the problem can be rewritten with the vector $\mathbf{\Phi'} \equiv (\phi_1,\ \phi_2,\ \ldots \phi_N)$ with N the number of hits. The matrix is then a (nonsymmetric) 3xN matrix. Let the original inverse matrix be defined as M with elements $M_{ij},\ i = 1, 3,\ j = 1, 3$. In terms of this, the elements of the new matrix are

$$M'_{ik} = M_{i1}\, r_k/\sigma_k^2 + M_{i2}\, r_k^2/\sigma_k^2 + M_{i3}\, r_k^3/\sigma_k^2$$

with $k = 1, N$ and N the number of hits used in the fit. The track parameters are found from $\mathbf{p} = \mathbf{M'\Phi'}$ which is simply

$$\begin{aligned} b &= \Sigma_k M'_{1k}\phi_k && (7)\\ \phi_0 &= \Sigma_k M'_{2k}\phi_k && (8)\\ \kappa &= \Sigma_k M'_{3k}\phi_k && (9)\\ & && (10) \end{aligned}$$

with $k = 1, N$ and N as above. The operation count increases by $3x(N-3)$ over the initial 3x3 form, but all terms in a given row of the matrix and all terms in the vector have the same scale. This significantly reduces the demand for high precision arithmatic. Further, a track-by-track rotation can be performed, such that at least one of the $\phi_k = 0$, reducing the number of operations.

Figure 1 shows the diference in impact parameter from a fit performed in a reduced precision floating point format and a fit using standard double precision floating point. If a floating point calculation is performed in programmable logic, reduced precision will be necessary to obtain sufficient performance. One sees clearly the improved numerical stability of the $3xN$ formulation. Such an improvement may also permit the calculation to be performed using integers, provided the matrix is precomputed and stored in a look up table.

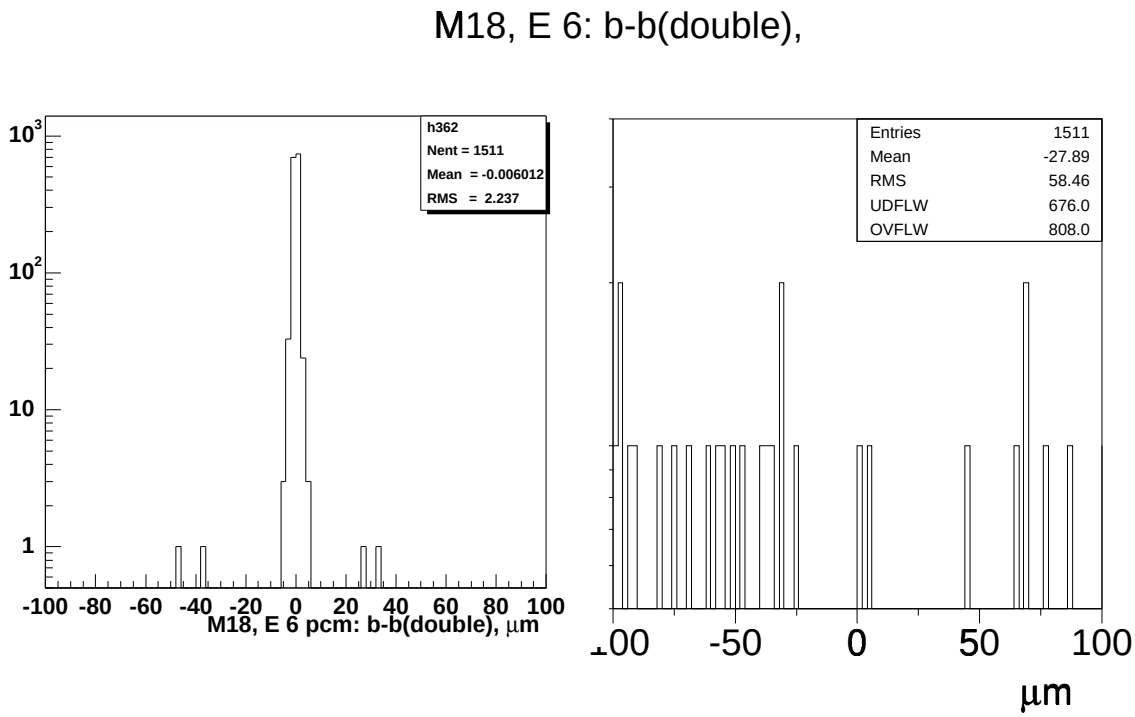

Figure 1: Two distributions of the difference in impact parameters resulting from a reduced precision fit and a standard double–precision fit. The left–hand panel is for a fit performed using the $3xN$ form of the linear algebra; the right–hand side is for the $3x3$ form. In both cases, the reduced precision floating point format has an 18–bit mantissa and a six bit exponent, and the matrix M was precomputed using full double precision. The superior numerical stability of the $3xN$ form of the problem is clear.